GEOMETRY OF FOUR DIMENSIONS.

By Henry P. Manning.

I have selected a few theorems and will give the proofs of
some of them in order that you may know how we study this
geometry and just how simple it is.*

I will suppose that we have started with *points* and regard all
figures as consisting of points. I assume a relation among
points which may be called the *collinear relation,* and define
lines as well as *planes* and *hyperplanes* by means of this relation.
The line is determined by two points, the plane by three points
not points of one line, and the hyperplane by four points not
points of one plane. The hyperplane, for example, consists of
the points that we get if we take four points not points of one
plane, all points collinear with any two of them, and all points
collinear with any two obtained by this process. Our space is a
hyperplane and geometry in a hyperplane is the ordinary solid
geometry.

Now in order to get a *space of four dimensions* it is only
necessary to assume the existence of five points not in one
hyperplane. Space of four dimensions consists of the points
that we get if we take five points not in one hyperplane, all
points collinear with any two of them, and all points collinear
with any two obtained by this process. For convenience I will
assume that all points lie in one space of four dimensions and
call this space *hyperspace.*

THEOREM I.—*If two points of a line lie in a hyperplane the
line lies entirely in the hyperplane, and if three non-collinear
points of a plane lie in a hyperplane the plane lies entirely in the
hyperplane.*

For the process of obtaining the line or plane is but a part of
the process of obtaining the hyperplane.

* These theorems are given with full details in the " Geometry of Four
Dimensions " soon to be published by The Macmillan Company.

THEOREM 2.—*One and only one hyperplane contains*

(1) *a plane and a point not in it, or an intersecting plane and line;*

(2) *two lines not in one plane;*

(3) *three lines through a point not in one plane;*

(4) *two planes intersecting in a line.*

From (4) it follows that two planes through a point but not in one hyperplane can have only one common point.

We might stand at this point and look through a tube and follow one plane completely around without seeing anything of the other plane, and then, turning in some direction on the second plane follow it completely around without seeing anything of the first. A plane extends in two directions, but there are two directions in which it does not extend. We can go completely around a plane just as in ordinary space we can go around a line. However, if we insist that we must see the figures of our geometry we shall stop right here. We can go on only with our reason. The theorem is incontestable, but the picture impossible.

On the other hand, the next three theorems will be readily accepted, although they are much more difficult to prove.

THEOREM 3.—*Two planes in a hyperplane with a point in common have a line in common.*

THEOREM 4.—*A plane and hyperplane with a point in common have a line in common.*

THEOREM 5.—*Two hyperplanes with a point in common have a plane in common.*

I will proceed to take up various kinds of perpendicularity.

A *line is perpendicular to a hyperplane* in much the same way that a line is perpendicular to a plane in solid geometry. We have three theorems:

THEOREM 6.—*The lines perpendicular to a given line m at a point O do not all lie in one plane.*

Proof.—Every point of hyperspace lies in a plane with m, and every plane containing m contains a perpendicular to m at O. If the perpendiculars were all in a plane α, every plane through m would contain two lines of the hyperplane determined by m and α, and so every point of hyperspace would be in this hyperplane.

THEOREM 7.—*A line m perpendicular at a point O to three lines not in one plane is perpendicular to every line through O in the hyperplane of the three lines.*

Proof.—Let *a*, *b*, and *c* be the three lines and let *d* be any other line through *O* in the hyperplane of these three. The plane *cd* intersects the plane *ab* in a line *h* (Th. 3). *m* is perpendicular to the plane *ab* and so to *h*, and then to the plane *hc* and so to *d*.

THEOREM 8.—*All the perpendiculars to a line at a point lie in a single hyperplane.*

Proof.—Let *m* be perpendicular to all the lines through *O* in a given hyperplane as in the preceding proof, and let *d* be given now as perpendicular to *m* at *O*. The plane *md* will intersect the hyperplane in a line *d'* (Th. 4), and in this plane we have *d* and *d'* both perpendicular to *m* at *O*. *d* must therefore coincide with *d'* and lie in the hyperplane.

A line and hyperplane are perpendicular at a point *O* when the line is perpendicular to all the lines of the hyperplane which pass through *O*.

THEOREM 9.—*A line perpendicular to a hyperplane at a point is perpendicular to all the planes of the hyperplane which pass through this point, and all the planes perpendicular to a line at a point lie in a perpendicular hyperplane.*

THEOREM 10.—*Through any point passes one and only one line perpendicular to a given hyperplane, or hyperplane perpendicular to a given line.*

THEOREM 11.—*Two lines perpendicular to a given hyperplane lie in a plane and are parallel.*

For they lie in a hyperplane (see Th. 2 (2)) and are both perpendicular to the plane in which their hyperplane intersects the given hyperplane.

In regard to planes we have three theorems somewhat like the three given to introduce perpendicular lines and hyperplanes (Ths. 6–8):

THEOREM 12.—*A plane has more than one line perpendicular to it at a given point.*

For the plane is the intersection of hyperplanes in each of which there is such a line.

THEOREM 13.—*Two lines perpendicular to a plane at a point O determine a second plane such that every line through O in either of these planes is perpendicular to every line through O in the other.*

THEOREM 14.—*All the perpendiculars to a plane at a given point lie in a single plane.*

Proof.—Let α and α' be two planes intersecting at O, every line of α through O perpendicular to every line of α' through O, and let d be a line through O perpendicular to α. α and d lie in a hyperplane (Th. 2 (1)) which intersects α' in a line d' (Th. 4), and in this hyperplane we have d and d' both perpendicular to α at O. d must therefore coincide with d' and lie in α'.

Two planes intersecting at a point O are *absolutely perpendicular* when every line of one through O is perpendicular to every line of the other through O.

THEOREM 15.—*Through any point passes one and only one plane absolutely perpendicular to a given plane.*

THEOREM 16.—*Two planes absolutely perpendicular to a third lie in a hyperplane and are parallel.*

Proof.—Let α and β be two planes absolutely perpendicular to a plane γ at points O and O', let c be the line OO', and let a and b be the lines through O and O' perpendicular to the hyperplane determined by α and c.* a lies in γ because a is perpendicular to α (Th. 14). b lies in γ because a and b lie in a plane (Th. 11) and γ is the only plane that contains a and O'. b is, then, perpendicular to β and β lies in the hyperplane determined by α and c (Th. 9). In this hyperplane we have α and β perpendicular to c and so parallel.

THEOREM 17.—*If two planes intersect in a line their absolutely perpendicular planes at any point O of their intersection intersect in a line.*

For they are both perpendicular to the line of intersection of the two given planes and so lie in the hyperplane perpendicular to this line at O (Ths. 9 and 3).

Two planes are perpendicular when they lie in a hyperplane and form right dihedral angles.

THEOREM 18.—*A plane perpendicular to one of two absolutely perpendicular planes at their point of intersection is perpendicular to the other.*

Proof.—Let α and α' be two planes absolutely perpendicular

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* It is not to be supposed that the figure shows how two planes absolutely perpendicular to a third look.

at a point O, and let β be perpendicular to α along a line which goes through O. Then β contains a line perpendicular to α at O, and so intersects α' in a line and lies in a hyperplane with α' (Ths. 14 and 2 (4)). Now the intersection of β and α, like all the lines of α through O, is perpendicular to α'. Therefore β, lying in a hyperplane with α', and containing a line perpendicular to α', is itself perpendicular to α'.

THEOREM 19.—*A plane intersecting two absolutely perpendicular planes in lines is perpendicular to both.*

THEOREM 20.—*Two planes intersecting in a line have at any point O of their intersection two common perpendicular planes, planes absolutely perpendicular to each other.*

Proof.—The two planes lie in a hyperplane (Th. 2 (4)) and their absolutely perpendicular planes at O intersect in a line and lie in a hyperplane. A common perpendicular plane is a plane which intersects all four of these planes in lines. Now a plane intersecting the two given planes in lines must lie in their hyperplane (Th. 1) or contain their line of intersection, and a plane intersecting in lines the planes absolutely perpendicular to the given planes must lie in their hyperplane or contain their line of intersection. A plane cannot lie in a hyperplane with two of these planes and contain the line of intersection of the other two, for the line is perpendicular to the hyperplane (Th. 7). Such a plane must, therefore, lie in both hyperplanes and be their plane of intersection, or contain both lines of intersection. The common perpendicular plane with which we are familiar, the plane in the hyperplane of the two given planes perpendicular to their intersection, is the plane of intersection of their hyperplane and the hyperplane of their absolutely perpendicular planes at O. The plane determined by the two lines of intersection is absolutely perpendicular to this plane (Th. 13), and these two are the only common perpendicular planes that the two given planes can have.

A *rectangular system* consists of four lines, six planes, and four hyperplanes. The lines are mutually perpendicular; the planes are three pairs of absolutely perpendicular planes, the two planes of any pair perpendicular to each of the four planes of the other two pairs. This system is the basis of rectangular coordinates.

A *plane and hyperplane are perpendicular* at a point *O* when the absolutely perpendicular plane lies in the hyperplane.

THEOREM 21.—*If a plane and hyperplane, intersecting in a line, are perpendicular at one point of their intersection, they are perpendicular all along this line.*

See proof of Th. 16.

We have several theorems like those in solid geometry about perpendicular planes; for example,

THEOREM 22.—*If a plane and hyperplane are perpendicular, any line in one perpendicular to their intersection is perpendicular to the other.*

THEOREM 23.—*If a plane and hyperplane are perpendicular, a plane in the hyperplane containing the intersection is perpendicular to the plane, and a plane in the hyperplane perpendicular to the intersection is absolutely perpendicular to the plane.*

Hyperplane angles are like ordinary dihedral angles. A *half-hyperplane* is that part of a hyperplane which lies on one side of any one of its planes, and the plane is the *face* of the half-hyperplane. A *hyperplane angle* is formed by two half-hyperplanes with a common face. A *plane angle* of a hyperplane angle is an angle with its sides in the two half-hyperplanes perpendicular to the face. The plane angle measures the hyperplane angle in the same way as with dihedral angles.

THEOREM 24.—*A hyperplane perpendicular to the face of a hyperplane angle intersects it in a dihedral angle with the same plane angle.*

A *plano-polyhedral angle* is formed by half-planes having a common edge and passing through the points of a plane polygon which does not lie in a hyperplane with the edge. The half-planes which pass through the vertices of the polygon are the *faces* of the plano-polyhedral angle. Two successive faces form a dihedral angle and the half-planes which pass through the points of a side of the polygon fill the interior of one of these dihedral angles. There are also hyperplane angles corresponding to the angles of the polygon.

THEOREM 25.—*A hyperplane intersecting the edge but not containing it intersects the plano-polyhedral angle in a polyhedral angle whose face angles and dihedral angles correspond respectively to the dihedral angles and hyperplane angles of of the plano-polyhedral angles.*

When the hyperplane is perpendicular to the edge the corresponding angles have the same measures. Thus we have many theorems analogous to the theorems of polyhedral angles. For example,

THEOREM 26.—*The sum of two dihedral angles of a plano-trihedral angle is greater than the third.*

We may think of a plano-polyhedral angle as generated by a polyhedral angle moving off in a direction away from its hyperplane.

A *polyhedroidal angle* is formed with a vertex and directing-polyhedron. It has face angles, dihedral angles, polyhedral angles, etc., corresponding to the edges, face angles, faces, etc., of the directing-polyhedron.

I come next to the angles of two planes. The planes and lines considered in this connection all pass through a point O. This will be left understood and not mentioned each time.

THEOREM 27.—*Given two planes, α and β, there is a half-line m of α whose angle with β is less than or equal to the angle made with β by any other half-line of α.*

The proof of this theorem involves the principle of continuity.

THEOREM 28.—*The plane of the minimum angle which a half-line of α makes with β is perpendicular also to α.*

Proof.—Let n be the projection of m upon β. Then if m were not the projection of n upon α, the line which is its projection would make with n, and so certainly with β, a smaller angle.*

The plane absolutely perpendicular to mn is another plane perpendicular to α and β (Th. 18). Two planes having a point in common always have two common perpendicular planes. The acute angles formed in these two planes are *the angles of the planes α and β*.

THEOREM 29.—*When the angles of two planes are equal the planes have an infinite number of common perpendicular planes on all of which they cut out the same angles, and any two of the common perpendicular planes cut out angles on β equal to the angles which they cut out on α.*

* This proof is given by C. J. Keyser, "Concerning the Angles and the Angular Determination of Planes in 4-space," *Bulletin of the American Mathematical Society,* Vol. 8, 1902, pp. 324–329.

When two planes have equal angles they are *isocline*. Their common perpendicular planes are also isocline. We have two *series of isocline planes, conjugate series,* all the planes of one series perpendicular to all the planes of the other series. There are two senses in which planes can be isocline; the planes of one of two conjugate series are isocline *positively* and the planes of the other series *negatively*. Isocline planes have many of the properties of parallel lines. For example,

THEOREM 30.—*If a plane (through O) intersects two isocline planes in lines the alternate interior dihedral angles are equal.**

I will mention some of the *hypersolids,* giving only a brief informal statement of a few of the theorems. The solids which go to form a hypersolid are *cells.*

A *hyperpyramid* has a polyhedron base. It may be cut from a polyhedroidal angle.

When the base is a tetrahedron the hyperpyramid is a pentahedroid and can be regarded as a hyperpyramid in five different ways. The five tetrahedrons can be cut apart sufficiently to be spread out in a hyperplane where we can see them all. Four of them rest upon the four faces of the fifth and can be folded on these faces and brought together again so as to enclose a portion of hyperspace.

A hyperpyramid with pyramid base can be regarded in two ways as such a hyperpyramid. It can also be regarded as having a vertex-edge and polygon base, and as cut from a plano-polyhedral angle by two hyperplanes. We will call it a *double pyramid.*

Similarly, we have *hypercones* and *double cones* cut from *hyperconical* and *plano-conical hypersurfaces,* a double cone being also in two ways a hypercone with a cone for base.

A *prismoidal hypersurface* is formed by a system of parallel lines (*elements*) with a directing-polyhedron. A *hyperprism* is cut from such a hypersurface by two parallel hyperplanes.

A *plano-prismatic hypersurface* is formed by a system of parallel planes (*plane elements*) with a directing-polygon, each element intersecting the plane of the polygon only in a single

* These properties of planes are developed by Stringham with the aid of quarternions, "On the Geometry of Planes in a Parabolic Space of Four Dimensions," *Transactions of the American Mathematical Society,* Vol. 2, 1901, pp. 183–214.

point.   The elements which contain the vertices of the polygon
are the *faces*, and the elements which lie between two successive
faces fill up a *cell*.   Any plane intersecting the elements only
in single points intersects the hypersurface in a polygon which
may be taken as directing-polygon.

When the elements of a plano-prismatic hypersurface inter-
sect the elements of a second plano-prismatic hypersurface only
in points the intersection of the two hypersurfaces is made up
in two ways of the lateral surfaces of a set of prisms.   There
are two sets of prisms, the prisms of one set being joined cross-
wise to the prisms of the other set.   These two sets of prisms
together with their interiors enclose a portion of hyperspace and
form a figure which may be called a *double prism*.   In a right
double prism where the elements of one hypersurface are abso-
lutely perpendicular to the elements of the other, the prisms of
either set can be cut apart from those of the other set and spread
out in a hyperplane so as to form a single prism with altitude
equal to the perimeter of any base of the prisms of the other set.
The directing-polygons of either hypersurface which lie in the
elements of the other are the *directing-polygons of the double
prism*.   Any two polygons intersecting in a single point and lying
in planes which have only this point in common are directing-
polygons of a double prism.   The double prism may be gen-
erated by moving each of these polygons with its interior around
the other, keeping it always parallel to itself.

A hyperprism whose bases are prisms can be regarded in two
ways as a hyperprism of this kind.   It can also be regarded as
a double prism, one set of prisms being parallelopipeds and the
other set a set of four.

Similarly, we have *hypercylindrical* and *plano-cylindrical
hypersurfaces,* and we have *hypercylinders, prism cylinders* and
double cylinders.   A prism cylinder is formed from a plano-
prismatic and a plano-cylindrical hypersurface, and a double
cylinder from two plano-cylindrical hypersurfaces.   A polygon
and a circle intersecting in a point and lying in planes which
have only this point in common can be used to generate a prism
cylinder, and two circles in the same way generate a double
cylinder.   Two circles in planes which are absolutely perpen-
dicular determine a double cylinder of double revolution.   The

surface of intersection of the two plano-cylindrical hypersur-
faces in this case is a surface of some importance in the theory
of complex variables.

A *hypersphere* is cut by a hyperplane in a sphere.   There are
great spheres and *small spheres*.   Any sphere in the hypersphere
has two *poles* and any circle has a *polar great circle*.   A great
circle is itself polar circle to its polar.

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